

# Averages of shifted convolution sums for $GL(3) \times GL(2)$

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**Abstract** Let  $A_f(1, n)$  be the normalized Fourier coefficients of a  $GL(3)$  Maass cusp form  $f$  and let  $a_g(n)$  be the normalized Fourier coefficients of a  $GL(2)$  cusp form  $g$ . Let  $\lambda(n)$  be either  $A_f(1, n)$  or the triple divisor function  $d_3(n)$ . It is proved that for any  $\epsilon > 0$ , any integer  $r \geq 1$  and  $r^{5/2}X^{1/4+7\delta/2} \leq H \leq X$  with  $\delta > 0$ ,

$$\frac{1}{H} \sum_{h \geq 1} W\left(\frac{h}{H}\right) \sum_{n \geq 1} \lambda(n) a_g(rn + h) V\left(\frac{n}{X}\right) \ll X^{1-\delta+\epsilon},$$

where  $V$  and  $W$  are smooth compactly supported functions, and the implied constants depend only on the associated forms and  $\epsilon$ .

**Keywords** Averages, shifted convolution sums,  $GL(3) \times GL(2)$

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## 1 Introduction

The shifted convolution sum problems have a long history in analytic number theory. Nontrivial bounds of various shifted convolution sums have been playing important roles in many central problems, such as quantum unique ergodicity, subconvexity and power moments of  $L$ -functions (see for example [1], [4], [6], [8], [12], [16], [22]). The first shifted convolution sum involving  $GL(3)$  Fourier coefficients was studied in [21] by Pitt who considered the shifted convolution sum of  $d_3(n)$  with the Fourier coefficients  $a_g(n)$  of a holomorphic cusp form  $g$ , where  $d_3(n) = \sum_{\substack{l_1 l_2 l_3 = n \\ l_j \in \mathbb{N}, j=1,2,3}} 1$  is the

triple divisor function which is the  $n$ -th coefficient of the cube of the Riemann zeta function  $\zeta^3(s)$ . Recently, Munshi [18] studied the general  $GL(3) \times GL(2)$  shifted convolution sum

$$\mathcal{D}_h(X) = \sum_{n \geq 1} A_f(1, n) a_g(n + h) V\left(\frac{n}{X}\right),$$

where  $A_f(1, n)$  are the Fourier coefficients of a  $GL(3)$  Maass cusp form  $f$ ,  $a_g(n)$  are those of a  $GL(2)$  Maass or holomorphic cusp form  $g$ ,  $1 \leq h \leq X^{1+\epsilon}$  an integer, and  $V$  is a smooth compactly supported function, and succeeded in showing that

$$\mathcal{D}_h(X) \ll_{f,g,\epsilon} X^{1-\frac{1}{20}+\epsilon}$$

by using the idea of factorizable moduli with the circle method of Jutila's version. As Munshi remarked in his paper, "it is expected that extra cancellation can be obtained by averaging over  $h$ ", which will be the main concern of this paper. In fact, we shall consider the following averages of  $GL(3) \times GL(2)$  shifted convolution sums

$$\mathcal{S}(H, X) = \frac{1}{H} \sum_{h \geq 1} W\left(\frac{h}{H}\right) \sum_{n \geq 1} \lambda(n) a_g(rn + h) V\left(\frac{n}{X}\right), \quad (1.1)$$

where  $W$  is another smooth compactly supported function,  $r \geq 1$  is an integer,  $\lambda(n)$  is either  $A_f(1, n)$  or  $d_3(n)$ . Here  $f$  is a Maass cusp form for  $SL(3, \mathbb{Z})$  and  $g$  is a Maass or holomorphic cusp form for  $SL(2, \mathbb{Z})$ . Our main result is the following theorem.

**Theorem 1** *For any  $\epsilon > 0$ , any integer  $r \geq 1$  and  $(rX)^{1/2+\epsilon} \leq H \leq X$ , we have*

$$\mathcal{S}(H, X) \ll X^{-A}$$

*for any  $A > 0$ . For any  $\epsilon > 0$ , any integer  $r \geq 1$  and  $r^{5/2}X^{1/4+7\delta/2} \leq H \leq (rX)^{1/2+\epsilon}$  with  $\delta > 0$ , we have*

$$\mathcal{S}(H, X) \ll X^{1-\delta+\epsilon}.$$

*Here the implied constants depend only on the associated forms and  $\epsilon$ .*

Recently, averages of shifted convolution sums for  $GL(2)$  cusp forms have been studied in [2], [14] and [23]. We note that for the shifted convolution sum in (1.1) without averaging and  $\lambda(n) = d_3(n)$ , Munshi's approach for  $\mathcal{D}_h(X)$  can also be applied (see [19]). Moreover, since  $d_3(n) \ll n^\epsilon$  for any  $\epsilon > 0$ , we can remove the smooth weight  $V$  in (1.1).

**Theorem 2** *Assume that  $a_g(n) \ll n^{\theta+\epsilon}$  for any  $\epsilon > 0$ . For any  $\epsilon > 0$ , any integer  $r \geq 1$  and  $(rX)^{1/2+\epsilon} \leq H \leq X$ , we have*

$$\frac{1}{H} \sum_{h \geq 1} W\left(\frac{h}{H}\right) \sum_{n \leq X} d_3(n) a_g(rn + h) \ll X^{-A}$$

*for any  $A > 0$ . For any  $\epsilon > 0$ , any integer  $r \geq 1$  and  $r^{5/2}X^{1/4+6\delta}(rX)^{5\theta/2} \leq H \leq (rX)^{1/2+\epsilon}$  with  $\delta > 0$ , we have*

$$\frac{1}{H} \sum_{h \geq 1} W\left(\frac{h}{H}\right) \sum_{n \leq X} d_3(n) a_g(rn + h) \ll_{g, \epsilon} X^{1-\delta+\epsilon}.$$

Note that we can take  $\theta = 0$  for  $g$  a holomorphic cusp form and  $\theta = 7/64$  for  $g$  a Maass cusp form (see [11]).

## 2 The circle method and Voronoi formulas

### 2.1 The circle method

As usual, denote  $\delta(n) = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$

**Lemma 1** ([7]) *For any  $P > 1$  there is a positive constant  $c_P$ , and a smooth function  $h(x, y)$  defined on  $(0, \infty) \times \mathbb{R}$ , such that*

$$\delta(n) = \frac{c_P}{P^2} \sum_{q=1}^{\infty} \sum_{c \bmod q}^* e\left(\frac{cn}{q}\right) h\left(\frac{q}{P}, \frac{n}{P^2}\right)$$

for  $n \in \mathbb{Z}$ . Here the  $*$  over the sum indicates that  $c$  and  $q$  are coprime. The constant  $c_P = 1 + O_A(P^{-A})$  for any  $A > 0$ . Moreover,  $h(x, y) \ll x^{-1}$  for all  $y$ , and  $h(x, y)$  is nonzero only when  $x \leq \max\{1, 2|y|\}$ . The smooth function  $h(x, y)$  satisfies

$$x^i \frac{\partial^i h}{\partial x^i}(x, y) \ll_i x^{-1} \quad \text{and} \quad \frac{\partial h}{\partial y}(x, y) = 0 \quad (2.1)$$

for  $x \leq 1$  and  $|y| \leq x/2$ . And also for  $|y| \geq x/2$ , we have

$$x^i y^j \frac{\partial^{i+j} h}{\partial x^i \partial y^j}(x, y) \ll_{i,j} x^{-1}. \quad (2.2)$$

We will apply Lemma 1 for larger  $H$  using the fact that we can choose  $P = \sqrt{Y}$  to detect the equation  $n = 0$  for integers in the range  $|n| \leq Y$ . For small  $H$ , Lemma 1 is not efficient to obtain savings (for small  $q$ ) in our problem and we will apply Jutila's variation of the circle method ([10])

which gives an approximation for  $I_{[0,1]}(x) = \begin{cases} 1, & x \in [0, 1], \\ 0, & \text{otherwise,} \end{cases}$  where  $I_S(x)$  is the characteristic

function of the set  $S$ . We have the following result (for a proof see [18], Lemma 4).

**Lemma 2** *Let  $\Omega \subset [1, Q]$ ,  $Q > 0$  and  $Q^{-2} \leq \eta \leq Q^{-1}$ . Define*

$$\tilde{I}_{\Omega, \eta}(x) = \frac{1}{2\eta L} \sum_{q \in \Omega} \sum_{c \bmod q}^* I_{\left[\frac{c}{q} - \eta, \frac{c}{q} + \eta\right]}(x), \quad (2.3)$$

where  $L = \sum_{q \in \Omega} \phi(q)$ . Then for any  $\epsilon > 0$ ,

$$\int_0^1 \left| 1 - \tilde{I}_{\Omega, \eta}(\beta) \right|^2 d\beta \ll \frac{Q^{2+\epsilon}}{\eta L^2}. \quad (2.4)$$

## 2.2 $GL(2)$ Voronoi formulas

For notational simplicity, we assume that  $g$  is a Hecke-Maass cusp form for  $SL(2, \mathbb{Z})$  with Laplace eigenvalue  $1/4 + \mu^2$  and normalized Fourier coefficients  $a_g(m)$ .

**Lemma 3** ([17]) *Let  $\psi(y) \in C_c^\infty(0, \infty)$ . For  $(c, q) = 1$ , we have*

$$\sum_{m=1}^{\infty} a_g(m) e\left(\frac{cm}{q}\right) \psi(m) = \frac{1}{q} \sum_{\pm} \sum_{m=1}^{\infty} a_g(\mp m) e\left(\pm \frac{\bar{c}m}{q}\right) \Psi^\pm\left(\frac{m}{q^2}\right),$$

where  $\bar{c}$  denote the multiplicative inverse of  $c \bmod q$ , and

$$\Psi^-(y) = -\frac{\pi}{\cosh(\pi\mu)} \int_0^\infty \psi(v) (Y_{2i\mu} + Y_{-2i\mu}) (4\pi\sqrt{yv}) dv, \quad (2.5)$$

$$\Psi^+(y) = 4 \cosh(\pi\mu) \int_0^\infty \psi(v) K_{2i\mu} (4\pi\sqrt{yv}) dv. \quad (2.6)$$

If  $\psi(y)$  is a smooth function of compact support in  $[AY, BY]$ , where  $Y > 0$  and  $B > A > 0$ , satisfying  $\psi^{(j)}(y) \ll_{A,B,j} Y^{-j}$  for any integer  $j \geq 0$ , then for any fixed  $\epsilon > 0$  and  $yY \gg Y^\epsilon$ ,  $\Psi^\pm(y)$  are negligibly small. For  $yY \ll Y^\epsilon$ , we have the trivial bound  $\Psi^\pm(y) \ll_{g,\epsilon} Y^{1+\epsilon}$ .

### 2.3 $GL(3)$ Voronoi formulas

Let  $f$  be a Hecke-Maass cusp form of type  $(\nu_1, \nu_2)$  for  $SL(3, \mathbb{Z})$  with normalized Fourier coefficients  $A_f(n_1, n_2)$ . Denote  $\mu_1 = -\nu_1 - 2\nu_2 + 1$ ,  $\mu_2 = -\nu_1 + \nu_2$ ,  $\mu_3 = \nu_1 + \nu_2 - 1$ . The generalized Ramanujan conjecture asserts that  $\text{Re}(\mu_j) = 0$ ,  $1 \leq j \leq 3$ , while the current record bound due to Luo, Rudnick and Sarnak [15] is  $|\text{Re}(\mu_j)| \leq \frac{1}{2} - \frac{1}{10}$ ,  $1 \leq j \leq 3$ .

Let  $\varphi(y)$  be a smooth function compactly supported on  $(0, \infty)$  and denote by  $\tilde{\varphi}(s)$  the Mellin transform of  $\varphi(y)$ . For  $k = 0, 1$ , we define

$$\Phi_k(y) := \int_{\text{Re}(s)=\sigma} (\pi^3 y)^{-s} \prod_{j=1}^3 \frac{\Gamma\left(\frac{1+s+\mu_j+2k}{2}\right)}{\Gamma\left(\frac{-s-\mu_j}{2}\right)} \tilde{\varphi}(-s-k) ds \quad (2.7)$$

with  $\sigma > \max_{1 \leq j \leq 3} \{-1 - \text{Re}(\mu_j) - 2k\}$ . Set

$$\Phi^\pm(y) = \Phi_0(y) \pm \frac{1}{i\pi^3 y} \Phi_1(y). \quad (2.8)$$

Then we have the following Voronoi formula.

**Lemma 4** ([5], [20]) *Let  $\varphi(y) \in C_c^\infty(0, \infty)$ . For  $(c, q) = 1$  we have*

$$\sum_{n \geq 1} A_f(1, n) e\left(\frac{cn}{q}\right) \varphi(n) = \frac{q\pi^{-\frac{5}{2}}}{4i} \sum_{\pm} \sum_{n_1|q} \sum_{n_2=1}^{\infty} \frac{A_f(n_2, n_1)}{n_1 n_2} S\left(\bar{c}, \pm n_2; \frac{q}{n_1}\right) \Phi^\pm\left(\frac{n_1^2 n_2}{q^3}\right),$$

where  $\bar{c}$  denote the multiplicative inverse of  $c \bmod q$  and  $S(m, n; c)$  is the classical Kloosterman sum.

Next we state the Voronoi formula for  $d_3(n)$  in Li's version (see [13]). Set  $\sigma_{0,0}(k, l) = \sum_{\substack{d_1|l \\ d_1 > 0}} \sum_{\substack{d_2|l \\ d_2 > 0, (d_2, k)=1}} 1$ . Let  $\gamma := \lim_{s \rightarrow 1} \left(\zeta(s) - \frac{1}{s-1}\right)$  be the Euler constant and  $\gamma_1 := -\frac{d}{ds} \left(\zeta(s) - \frac{1}{s-1}\right) \Big|_{s=1}$

be the Stieltjes constant. For  $\omega(y) \in C_c(0, \infty)$ ,  $k = 0, 1$  and  $\sigma > -1 - 2k$ , set

$$\Omega_k(y) = \frac{1}{2\pi i} \int_{\text{Re}(s)=\sigma} (\pi^3 y)^{-s} \frac{\Gamma\left(\frac{1+s+2k}{2}\right)^3}{\Gamma\left(\frac{-s}{2}\right)^3} \tilde{\omega}(-s-k) ds \quad (2.9)$$

with  $\tilde{\omega}(s) = \int_0^\infty \omega(u) u^{s-1} du$  the Mellin transform of  $\omega$ , and

$$\Omega^\pm(y) = \Omega_0(y) \pm \frac{1}{i\pi^3 y} \Omega_1(y). \quad (2.10)$$

**Lemma 5** *Let  $\omega(y) \in C_c^\infty(0, \infty)$ . For  $(c, q) = 1$  and  $c\bar{c} \equiv 1 \pmod{q}$  we have*

$$\begin{aligned}
& \sum_{n \geq 1} d_3(n) e\left(\frac{cn}{q}\right) \omega(n) \\
&= \frac{q}{2\pi^{\frac{3}{2}}} \sum_{\pm} \sum_{n|q} \sum_{m \geq 1} \frac{1}{nm} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0}\left(\frac{n}{n_1 n_2}, m\right) S\left(\pm m, \bar{c}; \frac{q}{n}\right) \Omega^\pm\left(\frac{mn^2}{q^3}\right) \\
&+ \frac{1}{2q^2} \tilde{\omega}(1) \sum_{n|q} n\tau(n) P_2(n, q) S\left(0, \bar{c}; \frac{q}{n}\right) \\
&+ \frac{1}{2q^2} \tilde{\omega}'(1) \sum_{n|q} n\tau(n) P_1(n, q) S\left(0, \bar{c}; \frac{q}{n}\right) \\
&+ \frac{1}{4q^2} \tilde{\omega}''(1) \sum_{n|q} n\tau(n) S\left(0, \bar{c}; \frac{q}{n}\right),
\end{aligned}$$

where  $P_1(n, q) = \frac{5}{3} \log n - 3 \log q + 3\gamma - \frac{1}{3\tau(n)} \sum_{d|n} \log d$ , and

$$\begin{aligned}
P_2(n, q) &= (\log n)^2 - 5 \log q \log n + \frac{9}{2} (\log q)^2 + 3\gamma^2 - 3\gamma_1 + 7\gamma \log n - 9\gamma \log q \\
&+ \frac{1}{\tau(n)} \left( (\log n + \log q - 5\gamma) \sum_{d|n} \log d - \frac{3}{2} \sum_{d|n} (\log d)^2 \right).
\end{aligned}$$

The functions  $\Phi^\pm(y)$  (also  $\Omega^\pm(y)$ ) have the following properties (see Sun [24] for proof).

**Lemma 6** *Suppose that  $\varphi(y)$  is a smooth function of compact support in  $[AY, BY]$ , where  $Y > 0$  and  $B > A > 0$ , satisfying  $\varphi^{(j)}(y) \ll_{A,B,j} P^j$  for any integer  $j \geq 0$ . Then for  $y > 0$  and any integer  $\ell \geq 0$ , we have*

$$\Phi^\pm(y) \ll_{A,B,\ell,\epsilon} (yY)^{-\epsilon} (PY)^3 \left(\frac{y}{P^3 Y^2}\right)^{-\ell}.$$

By Lemma 6, for any fixed  $\epsilon > 0$  and  $yY \gg Y^\epsilon (PY)^3$ ,  $\Phi^\pm(y)$  are negligibly small. For  $yY \ll Y^\epsilon (PY)^3$ , we can shift the contour of integration in (2.7) to  $\sigma = -3/5 + \epsilon$  with  $\epsilon > 0$  to get

$$\Phi^\pm(y) \ll (yY)^{\frac{3}{5}-\epsilon} PY. \quad (2.11)$$

### 3 Proof of Theorem 1

We write

$$\mathcal{S}(H, X) = \frac{1}{H} \sum_{h \geq 1} W\left(\frac{h}{H}\right) \sum_{n \geq 1} \lambda(n) V\left(\frac{n}{X}\right) \sum_{m \geq 1} a_g(m) \phi\left(\frac{m}{rX+h}\right) \delta(rn+h-m), \quad (3.1)$$

where  $\phi(y)$  is a smooth function compactly supported in  $[1/2, 5/2]$ , which equals 1 on  $[1, 2]$  and satisfies  $\phi^{(j)}(y) \ll_j 1$ . Taking  $P = \sqrt{6rX}$  and applying the circle method in Lemma 1, we have

$$\begin{aligned} \mathcal{S}(H, X) &= \frac{cP}{HP^2} \sum_{q \leq P} \sum_{c \bmod q}^* \sum_{n \geq 1} \lambda(n) e\left(\frac{crn}{q}\right) V\left(\frac{n}{X}\right) \sum_{m \geq 1} a_g(m) e\left(-\frac{cm}{q}\right) \\ &\quad \sum_{h \geq 1} e\left(\frac{ch}{q}\right) W\left(\frac{h}{H}\right) \phi\left(\frac{m}{rX+h}\right) h\left(\frac{q}{P}, \frac{rn+h-m}{P^2}\right). \end{aligned}$$

Applying Poisson summation to the  $h$ -sum (see Theorem 4.4 in [9]), we have

$$\begin{aligned} h\text{-sum} &= \sum_{\gamma \bmod q} e\left(\frac{c\gamma}{q}\right) \sum_{h \equiv \gamma \bmod q} W\left(\frac{h}{H}\right) \phi\left(\frac{m}{rX+h}\right) h\left(\frac{q}{P}, \frac{rn+h-m}{P^2}\right) \\ &= \frac{1}{q} \sum_{\gamma \bmod q} e\left(\frac{c\gamma}{q}\right) \sum_{h \in \mathbb{Z}} e\left(\frac{h\gamma}{q}\right) \int_{\mathbb{R}} W\left(\frac{x}{H}\right) \phi\left(\frac{m}{rX+x}\right) h\left(\frac{q}{P}, \frac{rn+x-m}{P^2}\right) e\left(-\frac{hx}{q}\right) dx \\ &= H \sum_{\substack{h \in \mathbb{Z} \\ h \equiv -c \bmod q}} \mathcal{I}(h, n, m, q), \end{aligned}$$

where

$$\mathcal{I}(h, n, m, q) = \int_{\mathbb{R}} W(x) \phi\left(\frac{m}{rX+Hx}\right) h\left(\frac{q}{P}, \frac{rn+Hx-m}{P^2}\right) e\left(-\frac{hHx}{q}\right) dx.$$

Note that the condition  $h \equiv -c \bmod q$  implies that  $(h, q) = 1$ . Then for  $h = 0$ , we have  $q = 1$ . For  $h \neq 0$ , by partial integration  $j$  times and (2.1)-(2.2), we have

$$\mathcal{I}(h, n, m, q) \ll_j \frac{P}{q} \left(\frac{|h|H}{q}\right)^{-j} \left(1 + \frac{H}{rX+H} + \frac{P}{q} \frac{H}{P^2}\right)^j \ll_j \frac{P}{q} \left(\frac{P}{H|h|}\right)^j.$$

Thus the contribution from  $|h| \geq P^{1+\epsilon}/H$  is negligible. In particular, if  $H > (rX)^{\frac{1}{2}+\epsilon}$ , we have

$$\begin{aligned} \mathcal{S}(H, X) &= \frac{cP}{P^2} \sum_{n \geq 1} \lambda(n) V\left(\frac{n}{X}\right) \sum_{m \geq 1} a_g(m) \mathcal{I}(0, n, m, 1) \\ &= \frac{cP}{P^2} \int_{\mathbb{R}} W(x) \sum_{n \geq 1} \lambda(n) V\left(\frac{n}{X}\right) \sum_{m \geq 1} a_g(m) \phi\left(\frac{m}{rX+Hx}\right) h\left(\frac{1}{P}, \frac{rn+Hx-m}{P^2}\right) dx \\ &\ll X^{-A} \end{aligned}$$

for any  $A > 0$ . Here we have used the fact of Booker [3] that for  $\pi$  an automorphic representation of  $GL_r(\mathbb{A}_{\mathbb{Q}})$  whose  $L$ -function  $L(s, \pi) = \sum_{n \geq 1} \lambda_{\pi}(n) n^{-s}$  is entire, and  $F$  a Schwartz function on  $(0, \infty)$ ,

$$\sum_{n \geq 1} \lambda_{\pi}(n) F\left(\frac{n}{X}\right) \ll_{\pi, A, F} X^{-A} \quad (3.2)$$

for any  $A > 0$ .

For  $H \leq (rX)^{\frac{1}{2}+\epsilon}$ , we write (3.1) as

$$\begin{aligned}\mathcal{S}(H, X) &= \frac{1}{H} \sum_{h \geq 1} W\left(\frac{h}{H}\right) \sum_{n \geq 1} \lambda(n) V\left(\frac{n}{X}\right) \sum_{m \geq 1} a_g(m) \phi\left(\frac{m}{rX+h}\right) \int_0^1 e((rn+h-m)\alpha) d\alpha \\ &= \frac{1}{H} \sum_{h \geq 1} W\left(\frac{h}{H}\right) \int_0^1 e(\alpha h) \sum_{n \geq 1} \lambda(n) e(\alpha rn) V\left(\frac{n}{X}\right) \sum_{m \geq 1} a_g(m) e(-\alpha m) \phi\left(\frac{m}{rX+h}\right) d\alpha.\end{aligned}$$

By Lemma 2 we shall approximate  $\mathcal{S}(H, X)$  by

$$\begin{aligned}\tilde{\mathcal{S}}(H, X) &= \frac{1}{H} \sum_{h \geq 1} W\left(\frac{h}{H}\right) \int_0^1 \tilde{I}_{\Omega, \eta}(\alpha) e(\alpha h) \sum_{n \geq 1} \lambda(n) e(\alpha rn) V\left(\frac{n}{X}\right) \\ &\quad \sum_{m \geq 1} a_g(m) e(-\alpha m) \phi\left(\frac{m}{rX+h}\right) d\alpha,\end{aligned}$$

where  $\tilde{I}_{\Omega, \eta}(\alpha)(x)$  is defined in (2.3). Then by Cauchy's inequality and (2.4),

$$\begin{aligned}\mathcal{S}(H, X) - \tilde{\mathcal{S}}(H, X) &\ll \frac{1}{H} \sum_{h \geq 1} W\left(\frac{h}{H}\right) \int_0^1 \left|1 - \tilde{I}_{\Omega, \eta}(\alpha)\right| \\ &\quad \left| \sum_{n \geq 1} \lambda(n) e(\alpha rn) V\left(\frac{n}{X}\right) \right| \left| \sum_{m \geq 1} a_g(m) e(-\alpha m) \phi\left(\frac{m}{rX+h}\right) \right| d\alpha \\ &\ll_{g, \epsilon} (rX)^{\frac{1}{2}+\epsilon} \left( \int_0^1 \left|1 - \tilde{I}_{\Omega, \eta}(\alpha)\right|^2 d\alpha \right)^{1/2} \\ &\quad \left( \int_0^1 \left| \sum_{n \geq 1} \lambda(n) e(\alpha rn) V\left(\frac{n}{X}\right) \right| d\alpha \right)^{1/2} \\ &\ll_{f, g, \epsilon} (rX)^\epsilon r^{1/2} X \left( \frac{Q^{2+\epsilon}}{\eta L^2} \right)^{1/2} \\ &\ll_{f, g, \epsilon} (rX)^\epsilon \frac{r^{1/2} X}{\sqrt{\eta} Q}\end{aligned}$$

since  $L \gg Q^{2-\epsilon}$ , where we have used the Rankin-Selberg estimate  $\sum_{n \leq Y} |A_f(1, n)|^2 \ll_{f, \epsilon} Y^{1+\epsilon}$  and the uniform bound in  $\alpha \in \mathbb{R}$

$$\sum_{m \geq 1} a_g(m) e(-\alpha m) \phi\left(\frac{m}{Y}\right) \ll_{g, \epsilon} Y^{1/2+\epsilon}.$$

Taking  $\eta = (rX + H)^{-1}$  we obtain

$$\mathcal{S}(H, X) = \tilde{\mathcal{S}}(H, X) + O\left((rX)^\epsilon \frac{rX^{\frac{3}{2}}}{Q}\right). \quad (3.3)$$

Then we only need to estimate  $\tilde{\mathcal{S}}(H, X)$ . Changing variable  $\alpha \rightarrow \frac{c}{q} + \beta$ , we have

$$\tilde{\mathcal{S}}(H, X) := \frac{1}{2\eta} \int_{-\eta}^{\eta} \tilde{\mathcal{S}}_{\beta}(H, X) d\beta,$$

where

$$\begin{aligned} \tilde{\mathcal{S}}_{\beta}(H, X) &= \frac{1}{HL} \sum_{q \in \Omega} \sum_{c \bmod q}^* \sum_{n \geq 1} \lambda(n) e\left(\frac{crn}{q}\right) V\left(\frac{n}{X}\right) e(\beta rn) \sum_{m \geq 1} a_g(m) e\left(-\frac{cm}{q}\right) e(-\beta m) \\ &\quad \sum_{h \geq 1} e\left(\frac{ch}{q}\right) W\left(\frac{h}{H}\right) \phi\left(\frac{m}{rX+h}\right) e(\beta h). \end{aligned} \quad (3.4)$$

Applying Poisson summation to the  $h$ -sum, we have

$$\begin{aligned} h\text{-sum} &= \sum_{\gamma \bmod q} e\left(\frac{c\gamma}{q}\right) \sum_{h \equiv \gamma \bmod q} W\left(\frac{h}{H}\right) \phi\left(\frac{m}{rX+h}\right) e(\beta h) \\ &= \frac{1}{q} \sum_{\gamma \bmod q} e\left(\frac{c\gamma}{q}\right) \sum_{h \in \mathbb{Z}} e\left(\frac{h\gamma}{q}\right) \int_{\mathbb{R}} W\left(\frac{x}{H}\right) \phi\left(\frac{m}{rX+x}\right) e(\beta x) e\left(-\frac{hx}{q}\right) dx \\ &= H \sum_{\substack{h \in \mathbb{Z} \\ h \equiv -c \bmod q}} I_{\beta}(h, m, q), \end{aligned}$$

where

$$I_{\beta}(h, m, q) = \int_{\mathbb{R}} W(x) \phi\left(\frac{m}{rX+Hx}\right) e(\beta Hx) e\left(-\frac{hHx}{q}\right) dx.$$

Now we choose the set of moduli  $\Omega$  as the prime set

$$\Omega = \{q : q \in [Q/2, Q] \text{ is prime and } (q, r) = 1\}.$$

Then the requirement  $L \gg_{\epsilon} Q^{2-\epsilon}$  is satisfied and  $h \neq 0$  since the condition  $h \equiv -c \bmod q$  implies that  $(h, q) = 1$ . By partial integration  $j$  times we have  $I_{\beta}(h, m, q) \ll_j (|h|H/q)^{-j}$  since  $|\beta| \leq \eta = (rX+H)^{-1}$ . Thus contribution from  $|h| \geq Q^{1+\epsilon}/H$  is negligible and

$$h\text{-sum} = H \sum_{\substack{1 \leq |h| \leq \frac{Q^{1+\epsilon}}{H} \\ h \equiv -c \bmod q}} I_{\beta}(h, m, q) + O((rX)^{-A}) \quad (3.5)$$

for any  $A > 0$ . Plugging (3.5) into (3.4), we need to estimate

$$\begin{aligned} &\frac{1}{L} \sum_{1 \leq |h| \leq \frac{Q^{1+\epsilon}}{H}} \sum_{\substack{q \in \Omega \\ (q, h) = 1}} \sum_{n \geq 1} \lambda(n) e\left(-\frac{hrn}{q}\right) V\left(\frac{n}{X}\right) e(\beta rn) \sum_{m \geq 1} a_g(m) e\left(\frac{hm}{q}\right) e(-\beta m) I_{\beta}(h, m, q) \\ &:= \int_{\mathbb{R}} W(x) e(\beta Hx) \tilde{\mathcal{S}}_{\beta, x}(H, X) dx, \end{aligned}$$



where

$$\begin{aligned}\tilde{\mathcal{S}}_{\beta,x}(H,X) &= \frac{1}{L} \sum_{1 \leq |h| \leq \frac{Q^{1+\epsilon}}{H}} \sum_{\substack{q \in \Omega \\ (q,h)=1}} e\left(-\frac{hHx}{q}\right) \sum_{n \geq 1} \lambda(n) e\left(-\frac{hrn}{q}\right) V\left(\frac{n}{X}\right) e(\beta rn) \\ &\quad \sum_{m \geq 1} a_g(m) e\left(\frac{hm}{q}\right) \phi\left(\frac{m}{rX+Hx}\right) e(-\beta m).\end{aligned}\tag{3.6}$$

We first apply  $GL(2)$  Voronoi formula in Lemma 3 to the  $m$ -sum in (3.6) to get

$$m\text{-sum} = \frac{1}{q} \sum_{\pm} \sum_{m=1}^{\infty} a_g(\mp m) e\left(\pm \frac{\bar{h}m}{q}\right) \Psi_{\beta,x}^{\pm}\left(\frac{m}{q^2}\right),\tag{3.7}$$

where  $\Psi_{\beta,x}^{\pm}(y)$  are defined in (2.5)-(2.6) with  $\psi(y) = \phi\left(\frac{y}{rX+Hx}\right) e(-\beta y)$ . Note that

$$\frac{d^j}{dy^j} \left\{ \phi\left(\frac{y}{rX+Hx}\right) e(-\beta y) \right\} \ll \left( \frac{1}{rX+H} + |\beta| \right)^j \ll \left( \frac{1}{rX} \right)^j.$$

Thus the contribution from  $|m| \gg Q^2(rX)^{\epsilon}/(rX)$  in (3.7) is negligible. For  $|m| \ll Q^2(rX)^{\epsilon}/(rX)$ , we have the trivial bound  $\Psi_{\beta,x}^{\pm}\left(\frac{m}{q^2}\right) \ll (rX)^{1+\epsilon}$ .

Next we want to apply the Voronoi formulas to the  $n$ -sum in (3.6).

**Case (i)**  $\lambda(n) = A_f(1, n)$ . We apply the  $GL(3)$  Voronoi formula in Lemma 4 to the  $n$ -sum in (3.6) to get

$$\begin{aligned}&\sum_{n \geq 1} A_f(1, n) e\left(-\frac{hrn}{q}\right) V\left(\frac{n}{X}\right) e(\beta rn) \\ &= \frac{q\pi^{-\frac{5}{2}}}{4i} \sum_{\pm} \sum_{n_1|q} \sum_{n_2=1}^{\infty} \frac{A_f(n_2, n_1)}{n_1 n_2} S\left(-\bar{h}r, \pm n_2; \frac{q}{n_1}\right) \Phi_{\beta}^{\pm}\left(\frac{n_1^2 n_2}{q^3}\right),\end{aligned}\tag{3.8}$$

where  $\Phi_{\beta}^{\pm}(y)$  are defined in (2.7)-(2.8) with  $\varphi(y) = V(y/X) e(\beta ry)$ . Note that  $\frac{d^j}{dy^j} \left\{ V\left(\frac{y}{X}\right) e(\beta ry) \right\} \ll X^{-j}$  for any  $j \geq 0$ . By Lemma 6, one sees that the contribution from  $n_1^2 n_2 \gg Q^3 X^{\epsilon}/X$  in (3.8) is negligible. For  $n_1^2 n_2 \ll Q^3 X^{\epsilon}/X$ , by (2.11) we get

$$\Psi_{\beta}^{\pm}\left(\frac{n_1^2 n_2}{q^3}\right) \ll_{f,\epsilon} \left(\frac{X n_1^2 n_2}{q^3}\right)^{\frac{3}{5}-\epsilon}.\tag{3.9}$$

By (3.6)-(3.9) and Weil's bound for Kloosterman sums, we conclude that

$$\begin{aligned}
\tilde{S}_{\beta,x}(H, X) &\ll_{f,g,\epsilon} \frac{1}{L} \sum_{\pm} \sum_{1 \leq |h| \leq \frac{Q^{1+\epsilon}}{H}} \sum_{q \in \Omega} \sum_{|m| \ll Q^2(rX)^\epsilon/(rX)} |a_g(\mp m)|(rX)^{1+\epsilon} \\
&\quad \sum_{n_1|q} \sum_{n_2 \ll Q^3 X^\epsilon/(n_1^2 X)} \frac{|A_f(n_2, n_1)|}{n_1 n_2} \left(\frac{q}{n_1}\right)^{1/2} \left(\frac{X n_1^2 n_2}{q^3}\right)^{\frac{3}{5}-\epsilon} \\
&\ll_{f,g,\epsilon} \frac{X^{\frac{3}{5}+\epsilon} Q}{H} \sum_{q \in \Omega} q^{-\frac{13}{10}} \sum_{n_1|q} n_1^{-\frac{3}{10}} \sum_{n_2 \ll Q^3 X^\epsilon/(n_1^2 X)} |A_f(n_2, n_1)| n_2^{-\frac{2}{5}} \\
&\ll_{f,g,\epsilon} \frac{(rX)^\epsilon Q^{5/2}}{H}, \tag{3.10}
\end{aligned}$$

where we have used the Rankin-Selberg estimates  $\sum_{|m| \ll N} |a_g(\mp m)| \ll_g N$  and  $\sum_{n_2 \leq N} |A_f(n_1, n_2)| \ll_f N|n_1|$ . Taking

$$Q = (rH)^{2/7} X^{3/7}.$$

Then for  $\lambda(n) = A_f(1, n)$  Theorem 1 follows from (3.3) and (3.10).

**Case (ii)**  $\lambda(n) = d_3(n)$ . Applying Lemma 5 to the  $n$ -sum in (3.6) we get

$$\begin{aligned}
&\sum_{n \geq 1} d_3(n) e\left(-\frac{hrn}{q}\right) V\left(\frac{n}{X}\right) e(\beta rn) \\
&= \frac{q}{2\pi^{\frac{3}{2}}} \sum_{\pm} \sum_{n|q} \sum_{l \geq 1} \frac{1}{nl} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0}\left(\frac{n}{n_1 n_2}, l\right) S\left(-\overline{hr}, \pm l; \frac{q}{n}\right) \Omega_{\beta}^{\pm}\left(\frac{n^2 l}{q^3}\right) \\
&\quad + \frac{1}{2q^2} \tilde{\omega}(1) \sum_{n|q} n \tau(n) P_2(n, q) \mu\left(\frac{q}{n}\right) + \frac{1}{2q^2} \tilde{\omega}'(1) \sum_{n|q} n \tau(n) P_1(n, q) \mu\left(\frac{q}{n}\right) \\
&\quad + \frac{1}{4q^2} \tilde{\omega}''(1) \sum_{n|q} n \tau(n) \mu\left(\frac{q}{n}\right), \tag{3.11}
\end{aligned}$$

where  $\Omega_{\beta}^{\pm}(y)$  are defined in (2.9)-(2.10) with  $\omega(y) = V(y/X) e(\beta ry)$ . As in the Case (i) the first term in (3.11) is essentially supported on  $n^2 l \ll Q^3 X^\epsilon/X$  and the contribution from the first term of (3.11) can be bounded similarly as that in the Case (i), which is at most  $\frac{(rX)^\epsilon Q^{5/2}}{H}$  with  $Q = (rH)^{2/7} X^{3/7}$ . For the remaining terms in (3.11), we have trivially

$$\tilde{\omega}^{(j)}(1) = \int_0^\infty \omega(u) (\log u)^j du \ll_j X (\log X)^j,$$

and they contribute (3.6) by

$$\frac{1}{L} \sum_{\pm} \sum_{1 \leq |h| \leq \frac{Q^{1+\epsilon}}{H}} \sum_{q \in \Omega} \frac{1}{q} \sum_{|m| \ll Q^2(rX)^\epsilon/(rX)} |a_g(\mp m)|(rX)^{1+\epsilon} \frac{X^{1+\epsilon}}{q^2} \sum_{n|q} n \tau(n) \log^2(nq) \ll (rX)^\epsilon \frac{X}{H}$$

which is  $\ll \frac{(rX)^\epsilon Q^{5/2}}{H}$  for  $Q = (rH)^{2/7} X^{3/7}$ . This finishes the proof of Theorem 1.

## 4 Proof of Theorem 2

By dyadic subdivisions we only need to estimate

$$\mathcal{T}^\sharp(H, Y) := \frac{1}{H} \sum_{h \geq 1} W\left(\frac{h}{H}\right) \sum_{Y < n \leq 2Y} d_3(n) a_g(rn + h),$$

where  $Y = 2^{-\ell}X$ ,  $1 \leq \ell \ll \log X$ ,  $\ell \in \mathbb{Z}$ . Note that

$$\mathcal{T}^\sharp(H, Y) = \frac{1}{H} \sum_{h \geq 1} W\left(\frac{h}{H}\right) \sum_{Y < n \leq 2Y} d_3(n) \sum_{m \geq 1} a_g(m) \phi\left(\frac{m}{rY + h}\right) \delta(rn + h - m),$$

where  $\phi(y)$  is as in Theorem 1, i.e., a smooth function compactly supported in  $[1/2, 5/2]$ , equals 1 on  $[1, 2]$  and satisfies  $\phi^{(j)}(y) \ll_j 1$ . Taking  $\mathcal{P} = \sqrt{4rY + 2H}$  and applying Lemma 1, we have

$$\begin{aligned} \mathcal{T}^\sharp(H, Y) &= \frac{c\mathcal{P}}{H\mathcal{P}^2} \sum_{q \leq \mathcal{P}} \sum_{c \bmod q}^* \sum_{Y < n \leq 2Y} d_3(n) e\left(\frac{crn}{q}\right) \sum_{m \geq 1} a_g(m) e\left(-\frac{cm}{q}\right) \\ &\quad \sum_{h \geq 1} e\left(\frac{ch}{q}\right) W\left(\frac{h}{H}\right) \phi\left(\frac{m}{rY + h}\right) h\left(\frac{q}{\mathcal{P}}, \frac{rn + h - m}{\mathcal{P}^2}\right). \end{aligned} \quad (4.1)$$

Then for  $H > (rX)^{\frac{1}{2}+\epsilon}$ , the proof of Theorem 2 is the similar as that of Theorem 1 by applying Poisson summation to the  $h$ -sum in (4.1) and using (3.2).

For  $H \leq (rX)^{\frac{1}{2}+\epsilon}$ , we let

$$\mathcal{T}(H, Y) := \frac{1}{H} \sum_{h \geq 1} W\left(\frac{h}{H}\right) \sum_{n \geq 1} d_3(n) a_g(rn + h) U\left(\frac{n}{Y}\right),$$

where  $U(y)$  is a smooth function compactly supported in  $[1, 2]$ , which equals 1 on  $[1 + \Delta^{-1}, 2 - \Delta^{-1}]$  ( $\Delta > 1$  is a parameter to be chosen optimally later) and satisfies  $U^{(j)}(y) \ll_j \Delta^j$ . Assume that  $a_g(n) \ll n^{\theta+\epsilon}$ . Then we have

$$\mathcal{T}^\sharp(H, Y) = \mathcal{T}(H, Y) + O_{g,\epsilon}\left(X\Delta^{-1}(rX)^{\theta+\epsilon}\right). \quad (4.2)$$

Note that

$$\mathcal{T}(H, Y) = \frac{1}{H} \sum_{h \geq 1} W\left(\frac{h}{H}\right) \sum_{n \geq 1} d_3(n) U\left(\frac{n}{Y}\right) \sum_{m \geq 1} a_g(m) \phi\left(\frac{m}{rY + h}\right) \int_0^1 e((rn + h - m)\alpha) d\alpha.$$

As in the proof of Theorem 1 we apply Lemma 2 to approximate  $\mathcal{T}(H, Y)$  by

$$\begin{aligned} \tilde{\mathcal{T}}(H, Y) &= \frac{1}{H} \sum_{h \geq 1} W\left(\frac{h}{H}\right) \int_0^1 \tilde{I}_{\Omega, \eta}(\alpha) e(\alpha h) \sum_{n \geq 1} d_3(n) e(\alpha rn) U\left(\frac{n}{Y}\right) \\ &\quad \sum_{m \geq 1} a_g(m) e(-\alpha m) \phi\left(\frac{m}{rY + h}\right) d\alpha, \end{aligned}$$

where  $\tilde{I}_{\Omega,\eta}(\alpha)(x)$  is defined in (2.3) with

$$\Omega = \{q : q \in [\mathcal{Q}/2, \mathcal{Q}] \text{ is prime and } (q, r) = 1\}.$$

Take  $\eta = (rY + H)^{-1}$ . Then by Cauchy's inequality and (2.4),

$$\mathcal{T}(H, Y) = \tilde{\mathcal{T}}(H, Y) + O\left((rX)^\epsilon \frac{rX^{\frac{3}{2}}}{\mathcal{Q}}\right). \quad (4.3)$$

In the following we estimate  $\tilde{\mathcal{T}}(H, Y)$ . We have

$$\tilde{\mathcal{T}}(H, Y) = \frac{1}{2\eta} \int_{-\eta}^{\eta} \tilde{\mathcal{T}}_{\beta}(H, Y) d\beta,$$

where

$$\begin{aligned} \tilde{\mathcal{T}}_{\beta}(H, Y) &= \frac{1}{HL} \sum_{q \in \Omega} \sum_{c \bmod q}^* \sum_{n \geq 1} d_3(n) e\left(\frac{crn}{q}\right) U\left(\frac{n}{Y}\right) e(\beta rn) \sum_{m \geq 1} a_g(m) e\left(-\frac{cm}{q}\right) e(-\beta m) \\ &\quad \sum_{h \geq 1} e\left(\frac{ch}{q}\right) W\left(\frac{h}{H}\right) \phi\left(\frac{m}{rY + h}\right) e(\beta h). \end{aligned} \quad (4.4)$$

As in Theorem 1, we apply Poisson summation to the  $h$ -sum to get

$$h\text{-sum} = H \sum_{\substack{1 \leq |h| \leq \frac{\mathcal{Q}^{1+\epsilon}}{H} \\ h \equiv -c \bmod q}} I_{\beta}(h, m, q) + O((rX)^{-A}) \quad (4.5)$$

for any  $A > 0$ . Plugging (4.5) into (4.4), we have

$$\begin{aligned} &\frac{1}{L} \sum_{1 \leq |h| \leq \frac{\mathcal{Q}^{1+\epsilon}}{H}} \sum_{\substack{q \in \Omega \\ (q, h) = 1}} \sum_{n \geq 1} d_3(n) e\left(-\frac{hrn}{q}\right) U\left(\frac{n}{Y}\right) e(\beta rn) \sum_{m \geq 1} a_g(m) e\left(\frac{hm}{q}\right) e(-\beta m) I_{\beta}(h, m, q) \\ &:= \int_{\mathbb{R}} W(x) e(\beta Hx) \tilde{\mathcal{T}}_{\beta, x}(H, Y) dx, \end{aligned}$$

where

$$\begin{aligned} \tilde{\mathcal{T}}_{\beta, x}(H, Y) &= \frac{1}{L} \sum_{1 \leq |h| \leq \frac{\mathcal{Q}^{1+\epsilon}}{H}} \sum_{\substack{q \in \Omega \\ (q, h) = 1}} e\left(-\frac{hHx}{q}\right) \sum_{n \geq 1} d_3(n) e\left(-\frac{hrn}{q}\right) U\left(\frac{n}{Y}\right) e(\beta rn) \\ &\quad \sum_{m \geq 1} a_g(m) e\left(\frac{hm}{q}\right) \phi\left(\frac{m}{rY + Hx}\right) e(-\beta m). \end{aligned} \quad (4.6)$$

Applying the  $GL(2)$  Voronoi formula in Lemma 2 to the  $m$ -sum in (4.6) we get

$$\begin{aligned} \tilde{\mathcal{T}}_{\beta, x}(H, Y) &= \frac{1}{L} \sum_{\pm} \sum_{1 \leq |h| \leq \frac{\mathcal{Q}^{1+\epsilon}}{H}} \sum_{\substack{q \in \Omega \\ (q, h) = 1}} \frac{1}{q} e\left(-\frac{hHx}{q}\right) \sum_{|m| \ll \mathcal{Q}^2(rY+H)^\epsilon/(rY+H)} a_g(\mp m) e\left(\pm \frac{\bar{h}m}{q}\right) \\ &\quad \Psi_{\beta, x}^{\pm}\left(\frac{m}{q^2}\right) \sum_{n \geq 1} d_3(n) e\left(-\frac{hrn}{q}\right) U\left(\frac{n}{Y}\right) e(\beta rn) + O_{g, \epsilon}(1). \end{aligned} \quad (4.7)$$

where  $\Psi_{\beta,x}^{\pm}(y)$  are defined in (2.5)-(2.6) with  $\psi(y) = \phi\left(\frac{y}{rY+Hx}\right)e(-\beta y)$ . and satisfy  $\Psi_{\beta,x}^{\pm}\left(\frac{m}{q^2}\right) \ll (rY+H)^{1+\epsilon}$ .

Next we apply the Voronoi formula for  $d_3(n)$  in Lemma 5 to the  $n$ -sum in (4.7) to get (3.11) with  $V\left(\frac{n}{X}\right)$  replaced by  $U\left(\frac{n}{Y}\right)$  and  $\omega(y) = U\left(\frac{y}{Y}\right)e(\beta ry)$ . By Weil's bound for Kloosterman sums, the contribution from the last three terms in (3.11) to (4.7) is at most

$$\begin{aligned} & \frac{1}{L} \sum_{\pm} \sum_{1 \leq |h| \leq \frac{Q^{1+\epsilon}}{H}} \sum_{\substack{q \in \Omega \\ (q,h)=1}} \frac{1}{q} \sum_{|m| \ll Q^2(rY+H)^{\epsilon}/(rY+H)} |a_g(\mp m)| \frac{(rY+H)^{1+\epsilon} Y}{q^2} \sum_{n|q} n \tau(n) (\log nq)^2 \\ & \ll_{g,\epsilon} (rX)^{\epsilon} \frac{X}{H}. \end{aligned} \quad (4.8)$$

For the first term in (3.11), we note that  $\frac{d^j}{dy^j} \{U\left(\frac{y}{Y}\right)e(\beta ry)\} \ll_{g,\epsilon} \left(\frac{\Delta}{Y}\right)^j$  for any  $j \geq 0$ . By Lemma 6, the contribution from  $n^2 l \gg q^3 \Delta^3 (qY)^{\epsilon}/Y$  is negligible. For  $n^2 l \ll q^3 \Delta^3 Y^{\epsilon}/Y$ , we shift the contour of integration in (2.9) to  $\sigma = -1/2 - \epsilon$  with  $\epsilon > 0$  to get

$$\Omega_{\beta}^{\pm}\left(\frac{n^2 l}{q^3}\right) \ll_{\epsilon} \Delta \left(\frac{Y n^2 l}{q^3}\right)^{1/2+\epsilon}. \quad (4.9)$$

By (4.9) and Weil's bound for Kloosterman sums, one sees that the first term in (3.11) contributes  $\tilde{\mathcal{T}}_{\beta,x}(H, Y)$  in (4.7) by

$$\begin{aligned} & \frac{1}{L} \sum_{\pm} \sum_{1 \leq |h| \leq \frac{Q^{1+\epsilon}}{H}} \sum_{\substack{q \in \Omega \\ (q,h)=1}} \sum_{|m| \ll Q^2(rY+H)^{\epsilon}/(rY+H)} |a_g(\mp m)| (rY+H)^{1+\epsilon} \\ & \sum_{n|q} \sum_{l \ll q^3 \Delta^3 Y^{\epsilon}/(n^2 Y)} \frac{1}{nl} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0}\left(\frac{n}{n_1 n_2}, l\right) \left(\frac{q}{n}\right)^{1/2} \left(\frac{Y n^2 l}{q^3}\right)^{1/2+\epsilon} \\ & \ll_{g,\epsilon} \frac{(rY+H)^{\epsilon} Q}{H} \sum_{Q/2 \leq q \leq Q} \sum_{n|q} \sum_{l \ll q^3 \Delta^3 (qY)^{\epsilon}/(n^2 Y)} \frac{d_3(l) d_3(n)}{nl} \left(\frac{q}{n}\right)^{1/2} \left(\frac{Y n^2 l}{q^3}\right)^{1/2} \\ & \ll_{g,\epsilon} \frac{(rY+H)^{\epsilon} Q Y^{1/2} \Delta}{H} \sum_{Q/2 \leq q \leq Q} q^{-1} \sum_{n|q} n^{-1/2} \sum_{l \ll q^3 \Delta^3 (qY)^{\epsilon}/(n^2 Y)} l^{-1/2} \\ & \ll_{g,\epsilon} (rX)^{\epsilon} \frac{Q^{5/2} \Delta^{5/2}}{H}. \end{aligned} \quad (4.10)$$

By (4.2), (4.3), (4.8) and (4.10) we take  $Q = (rH)^{2/7} X^{3/7} \Delta^{-5/7}$  and obtain

$$\mathcal{T}^{\sharp}(H, Y) \ll_{g,\epsilon} \frac{(rH\Delta)^{5/7} X^{15/14}}{H} + \frac{X(rX)^{\theta+\epsilon}}{\Delta}.$$

Then Theorem 2 follows by choosing  $\Delta = (HX)^{1/6} (rX)^{7/(12\theta)} (r^2 X)^{-5/24}$ .

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